

# Kähler-Ricci Flow with Degenerate Initial Class

Zhou Zhang <sup>\*</sup>  
University of Michigan  
at Ann Arbor

## Abstract

In [2], the weak Kähler-Ricci flow was introduced for various geometric motivations. In the current work, we take further consideration on setting up the weak flow. Namely, the initial class is allowed to be no longer Kähler. The convergence as  $t \rightarrow 0^+$  is of great importance to study for this topic.

## 1 Motivation and Set-up

Kähler-Ricci flow, the complex version of Ricci flow, has been under intensive study over the past twenty some years. In [10] and more recently [9], G. Tian proposed the intriguing program of constructing globally existing (weak) Kähler-Ricci flow with canonical (singular) limit at infinity and applying it to the study of general algebraic manifold.

Generally speaking, one should expect the classic Kähler-Ricci flow to encounter singularity at some finite time which is completely decided by cohomology information according to the optimal existence in [11]. Just as what people wanted to do and have had successes in some cases for Ricci flow, surgery on the underlying manifold should be expected. For Kähler-Ricci flow, we naturally want the surgery to have more flavor in algebraic geometry. For surface of general type, we only need the blow-down of  $(-1)$ -curves to apply the construction in [2] to push the flow through finite time singularities. The degenerate class at the singularity time would become Kähler for the new manifold because the  $(-1)$ -curves causing the cohomology degeneration have been crushed to points. Things can get significantly more complicated for higher dimensional manifold. In (complex) dimension 3, flips are involved. Simply speaking, one needs to blow up the manifold and then blow down. Naturally, we could expect the transformation of the degenerate class is still not Kähler. In this note, we want to say this is not a problem if formally the Kähler-Ricci flow is instantly taking the class into the Kähler cone of the new manifold. As in [2], short time existence is the topic. In the following, the precise problem under consideration is stated

---

<sup>\*</sup>Research supported in part by National Science Foundation grants DMS-0904760.

and we set up the a priori weak flow following the same idea as in the previous work.

Let  $X$  be a closed Kähler manifold with  $\dim_{\mathbb{C}} X = n \geq 2$ . We consider the following version of Kähler-Ricci flow over  $X$ ,

$$\frac{\partial \tilde{\omega}_t}{\partial t} = -\text{Ric}(\tilde{\omega}_t) - \tilde{\omega}_t, \quad \tilde{\omega}_0 = \omega_0. \quad (1.1)$$

More importantly, the weak initial data

$$\omega_0 = \omega + \sqrt{-1} \partial \bar{\partial} v$$

where  $\omega$  is a real, smooth, and closed  $(1, 1)$ -form with  $[\omega]$  being nef. (i.e. numerically effective, in other words, on the boundary of the Kähler cone of  $X$ ), and  $v \in PSH_{\omega}(X) \cap L^{\infty}(X)$  where  $v \in PSH_{\omega}(X)$  means  $v + \sqrt{-1} \partial \bar{\partial} v \geq 0$  weakly (i.e. in the sense of distribution).

**Remark 1.1.** *The choice of Kähler-Ricci in this version is not essential. Our discussion is even valid for other unconventional Kähler-Ricci type of flows as in [11].*

The main additional assumption for this work is the following.

**Suppose  $[\omega_0] = [\omega]$  is on the boundary of the Kähler cone of  $X$  (in the cohomology space  $H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$ ). The ray starting from  $[\omega]$  and in the direction towards the canonical class of  $X$ ,  $K_X$  enters the Kähler cone instantly.**

Clearly, there is no need for  $K_X$  to be Kähler for this to happen.

There are other motivations to study this case besides defining weak flow to realize Tian's program as mentioned before. In general, there is this understanding that the existence of Kähler-Ricci flow to be decided completely by the cohomology information from formal ODE consideration. The main theorem below would strengthen this philosophic point of view.

**Theorem 1.2.** *In the above setting, one can define a unique weak Kähler-Ricci flow from the approximation construction which becomes smooth instantly and satisfying (1.1).*

Formally, we see  $[\tilde{\omega}_t] = [\omega_t] \in H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$  where

$$\omega_t = \omega_{\infty} + e^{-t}(\omega - \omega_{\infty})$$

with  $\omega_{\infty} = -\text{Ric}(\Omega) := -\sqrt{-1} \partial \bar{\partial} \log \left( \frac{\Omega}{(\sqrt{-1})^n dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n} \right)$  in a local coordinate system  $\{z_1, \dots, z_n\}$ . The main additional assumption above simply means

$$[\omega_t] = e^{-t}[\omega] + (1 - e^{-t})K_X$$

is Kähler for  $t \in (0, T)$  for some  $T > 0$ . It is now routine to see at least formally (1.1) would be equivalent to the following evolution equation for a space-time function  $u$ ,

$$\frac{\partial u}{\partial t} = \log \frac{(\omega_t + \sqrt{-1} \partial \bar{\partial} u)^n}{\Omega} - u, \quad u(\cdot, 0) = v \quad (1.2)$$

with the understanding of  $\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$ .

Just as for classic smooth Kähler-Ricci flow, we'll focus on defining weak version of (1.2) instead of (1.1). Their equivalence in the category of smooth objects would make the weak version for (1.2) naturally the weak version for (1.1) in sight of the smoothing effect in Theorem 1.2.

It's time to describe the **approximation construction** mentioned in Theorem 1.2, which is similar to what has been applied in [2] except that now  $[\omega]$  is no longer Kähler. The idea is to find approximation of the initial data, use them as initial data to get a sequence of flows and finally take limit of the flows. The detail is as follows.

Take some Kähler metric  $\omega_1$  over  $X$ . For any  $\epsilon \geq 0$ , set  $\omega(\epsilon) = \omega + \epsilon\omega_1$  and  $\omega_t(\epsilon) = \omega_\infty + e^{-t}(\omega(\epsilon) - \omega_\infty)$ . Using the regularization result in [1], one has for any sequence  $\{\epsilon_j\}$  decreasing to 0 as  $j \rightarrow \infty$ , a sequence of functions  $\{v_j\}$  with  $v_j \in C^\infty(X)$  and  $\omega(\epsilon_j) + \sqrt{-1}\partial\bar{\partial}v_j > 0$ , decreasing to  $v$  accordingly. Then one considers the Kähler-Ricci flows,

$$\frac{\partial \tilde{\omega}_t(\epsilon_j)}{\partial t} = -\text{Ric}(\tilde{\omega}_t(\epsilon_j)) - \tilde{\omega}_t(\epsilon_j), \quad \tilde{\omega}_0(\epsilon) = \omega(\epsilon_j) + \sqrt{-1}\partial\bar{\partial}v_j. \quad (1.3)$$

At the level of metric potential, we have

$$\frac{\partial u_j}{\partial t} = \log \frac{(\omega_t(\epsilon_j) + \sqrt{-1}\partial\bar{\partial}u_j)^n}{\Omega} - u_j, \quad u_j(\cdot, 0) = v_j. \quad (1.4)$$

They are in the classic setting of Kähler-Ricci flow. By choosing the  $T$  (before (1.2)) properly, all these flows for  $j \gg 1$  (i.e.  $\epsilon_j$  sufficiently small) would exists for  $t \in [0, T)$  from cohomology consideration <sup>1</sup>.

In sight of  $v_j$  and  $\omega_t(\epsilon_j)$  decreasing to  $v$  and  $\omega_t$  as  $j \rightarrow \infty$ . Applying Maximum Principle, one can see  $u_j$  is also decreasing as  $j \rightarrow \infty$ . In principle, this would allow us to take and get a limit for each  $t \in [0, T)$ ,  $u(\cdot, t) \in PSH_{\omega_t}(X)$ , which is the weak flow wanted. For this to be true literally, one needs to make sure for each such  $t$ , the decreasing limit of  $u_j(\cdot, t)$  won't be  $-\infty$  uniformly over  $X$ . At the initial time, this is obviously the case. For  $t \in (0, T)$ , this would be the case as seen in later by applying Kolodziej's  $L^\infty$ -estimate (as in [5]) <sup>2</sup>.

Of course, one needs to make sure the limit, which is at this moment just a family of positive  $(1, 1)$ -current with parameter  $t$ , is more classic. This clearly boils down to obtain uniform estimates for  $u_j$ .

Before heading into the search for those uniform estimates, let's justify the uniqueness statement of Theorem 1.2. This is again a fairly direct application of Maximum Principle.

To begin with, let's observe that the decreasing convergence of  $v_j \rightarrow v$  can be strengthened to strictly decreasing convergence without changing the limit. Suppose  $\{v_j\}$  is only a decreasing sequence, then  $\{v_j + \frac{1}{j}\}$  is strictly decreasing

<sup>1</sup>As in [2], it's the short time existence that we are interested in, i.e. the small interval near  $t = 0$ .

<sup>2</sup>One can also achieve this using the classic PDE argument involving Moser Iteration.

with the same limit. Clearly, the affect on the solution of (1.4),  $u_j$  is negligible as  $j \rightarrow \infty$ . Also, the decreasing limit won't be affected by taking subsequence.

Secondly, we see the choice of sequence  $\{\epsilon_j\}$  won't affect the limit (i.e. the weak flow). Let's take two strictly decreasing sequences,  $\{v_j\}$  and  $\{v_\alpha\}$  in the construction before (1.3). Since for each  $j$ ,  $v_j > v$  and  $v_\alpha$  decreases to  $v$ , by Dini's Theorem,  $v_\alpha < v_j$  for  $\alpha$  sufficiently large. The other direction is also right. So by taking subsequences, still denoted by  $\{v_{j_a}\}$  and  $\{v_{\alpha_b}\}$ , we have

$$\cdots < v_{\alpha_2} < v_{j_2} < v_{\alpha_1} < v_{j_1}.$$

Applying Maximum Principle to (1.4), one has

$$\cdots < u_{\alpha_2} < u_{j_2} < u_{\alpha_1} < u_{j_1},$$

and so they have the same limit.

Now we take care of the general case. In the construction before (1.3), suppose we have chosen different  $\omega_1$ 's with different strictly decreasing sequences  $\{v_j\}$ . Then we can use the argument in the above situation and also make sure the corresponding  $\omega_t(\epsilon_j)$ 's have the same comparison relation. Thus Maximum Principle would still give the same kind of comparison for solutions of (1.4). Hence the limit would still be the same.

**Remark 1.3.** *It's not hard to make this construction even more flexible, and so far, this is pretty much the only way to come up with a reasonable weak flow. So even without any description of the situation as  $t \rightarrow 0^+$ , it is not so artificial to call this a weak flow initiating from  $\omega_0 = \omega + \sqrt{-1}\partial\bar{\partial}v$ .*

## 2 Local General Estimates

Now we begin the search for estimates uniform for all approximation flows (i.e.  $\epsilon > 0$  where we have such a flow). For simplicity, we'll **omit  $j$  and  $\epsilon_j$  in the notations** below which would unfortunately make (1.3) and (1.4) look exactly like (1.1) and (1.2) respectively. However this would help us keeping in mind about the degeneracy of the background form, and so is not such a terrible choice considering what we are trying to do.

**Note:**  $C$  below would stand for fixed positive constant which might be different at places. Its dependence on other constants should be clear from the context.

Clearly,  $u \leq C$  by Maximum Principle for (1.4). This is for all time. For the other estimates, the idea is trying to eliminate the affect of the initial data as completely as possible because we don't have control of the initial data except for  $L^\infty$ -bound of the potential. Also recall that we only need estimates for short time.

Notice that in (1.4), the initial value of the background form,  $\omega$  might not be non-negative. One can actually make better use of that  $\omega + \sqrt{-1}\partial\bar{\partial}v > 0$  by

looking at the following evolution at the level of metric potential for the same flow (1.3),

$$\frac{\partial \phi}{\partial t} = \log \frac{(\hat{\omega}_t + \sqrt{-1}\partial\bar{\partial}\phi)^n}{\Omega} - \phi, \quad \phi(\cdot, 0) = 0, \quad (2.1)$$

where  $\hat{\omega}_t = \omega_\infty + e^{-t}(\omega + \sqrt{-1}\partial\bar{\partial}v - \omega_\infty)$ . It's easy to see the relation between the solutions of (1.4) and (2.1) is  $u = \phi + e^t \cdot v$ . So  $\phi \leq C$ , which is not clear by applying Maximum Principle to (2.1) because of the lack of uniform control for  $\omega + \sqrt{-1}\partial\bar{\partial}v$  as form.

This following equation is obtained by taking  $t$ -derivative of (2.1) and making some transformations

$$\frac{\partial}{\partial t} \left( (e^t - 1) \frac{\partial \phi}{\partial t} - \phi \right) = \Delta_{\tilde{\omega}_t} \left( (e^t - 1) \frac{\partial \phi}{\partial t} - \phi \right) + n - \langle \tilde{\omega}_t, \omega + \sqrt{-1}\partial\bar{\partial}v \rangle.$$

Since  $\omega + \sqrt{-1}\partial\bar{\partial}v > 0$ , applying Maximum Principle and noticing the lower bound of the initial value and the uniform upper bound of  $\phi$ , we have for  $t \in (0, T)$ ,

$$\frac{\partial \phi}{\partial t} \leq \frac{C}{e^t - 1},$$

which gives the following bound of  $\frac{\partial u}{\partial t}$  since  $\frac{\partial u}{\partial t} = \frac{\partial \phi}{\partial t} + e^t \cdot v$ ,

$$\frac{\partial u}{\partial t} \leq \frac{C}{e^t - 1}.$$

For any  $t \in (0, T)$ , we have the background form  $[\omega_t(\epsilon)]$  being uniformly Kähler, i.e. the small interval corresponding to  $\epsilon$  is in the Kähler cone. Together with the above upper bound for  $\frac{\partial u}{\partial t}$ , one can apply Kołodziej's  $L^\infty$ -estimate (as in [5]) for (1.4) in the form of

$$(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n = e^{\frac{\partial u}{\partial t} + u} \Omega$$

to achieve the  $L^\infty$ -bound for  $u$ . So now we also have  $u(\cdot, t) \geq -C(t)$  with  $C(t)$  finite for  $t \in (0, T)$ . In fact, we know by the result in [6] that  $u(\cdot, t)$  is Hölder continuous for these  $t$ 's<sup>3</sup>.

**Remark 2.1.** *The original results on  $L^\infty$ -estimate (as in [5], [11] and [12]) are usually stated for Monge-Ampère equation in the form  $(\omega + \sqrt{-1}\partial\bar{\partial}u)^n = f \cdot \Omega$  where  $[\omega]$  might be degenerate,  $f \geq 0$  is in some  $L^{p>1}$ -space and  $\Omega$  is a (non-degenerate) smooth volume form. There are more than one way to translate this when applying to the equation with  $e^u$  on the right hand side.*

*Method I: get  $L^p$  bound for the measure  $f \cdot e^u \Omega$ , then one knows the normalized  $u$  would be bounded from the original result. In other words,  $u$  only takes value in some interval with well controlled length. Then the upper bound of  $u$ , which usually comes from direct Maximum Principle argument, together with the upper bound for  $f \cdot \Omega$  which guarantees  $u$  can not take too small value all*

---

<sup>3</sup>The Hölder exponent will also depend on  $t$ .

over in sight of the total volume having a lower bound, would provide the bound for  $u$  itself.

*Method II:* get  $L^p$  bound for the measure  $f \cdot \Omega$ , then consider the equation  $(\omega + \sqrt{-1}\partial\bar{\partial}w)^n = Cf \cdot \Omega$ . The idea is to apply Maximum Principle to the quotient of these two equations. In order to control the (normalized) solution  $w$  for this auxiliary equation, one needs to control the constant  $C$  (from above), which means we need a lower bound for the total volume for the measure  $f \cdot \Omega$ . Again this can be achieved from the upper bound for  $u$ .

It's not hard to see that these two methods are merely different combinations of the same set of information.

Since for any  $t \in [\lambda_1, \lambda_2] \subset (0, T)$ ,  $[\omega_t]$  is uniformly Kähler (for any approximation flow), by properly choosing  $\Omega$  (and so  $\omega_\infty$ )<sup>4</sup>, one has  $\omega_t$  being uniform as Kähler metric for  $t \in [\lambda_1, \lambda_2]$ <sup>5</sup>. Clearly,  $(0, T)$  can be exhausted by such closed intervals. Of course, we only care for the end towards  $t = 0$ .

Now let's translate the time to make  $\lambda_1$  the new initial time. From the discussion before, we have uniform bounds from both sides for  $u$  and the uniform upper bound for  $\frac{\partial u}{\partial t}$ . By taking  $t$ -derivative for (1.4) and making some transformations, we have the two equations below

$$\frac{\partial}{\partial t} \left( (e^t - 1) \frac{\partial u}{\partial t} \right) = \Delta_{\tilde{\omega}_t} \left( (e^t - 1) \frac{\partial u}{\partial t} \right) - (1 - e^{-t}) \langle \tilde{\omega}_t, \omega - \omega_\infty \rangle + \frac{\partial u}{\partial t}, \quad (2.2)$$

$$\frac{\partial}{\partial t} \left( (e^t - 1) \frac{\partial u}{\partial t} - u \right) = \Delta_{\tilde{\omega}_t} \left( (e^t - 1) \frac{\partial u}{\partial t} - u \right) + n - \langle \tilde{\omega}_t, \omega \rangle. \quad (2.3)$$

Notice that the  $\omega$  here is indeed  $\omega_{\lambda_1}$  and is uniform as Kähler metric for all approximation flows.

In the small time interval  $[0, \lambda_2 - \lambda_1]$  (after translation), we have made sure that  $\omega_t > 0$ , which is

$$\omega_\infty + e^{-t}(\omega - \omega_\infty) = \omega - (1 - e^{-t})(\omega - \omega_\infty) > 0,$$

so one can choose  $\delta \in (0, 1)$  such that for these  $t$ 's,

$$\delta\omega - (1 - e^{-t})(\omega - \omega_\infty) > 0.$$

Use this  $\delta$  to multiply (2.3) and take difference with (2.2) to arrive at

$$\begin{aligned} \frac{\partial}{\partial t} \left( (1 - \delta)(e^t - 1) \frac{\partial u}{\partial t} + \delta u \right) &= \Delta_{\tilde{\omega}_t} \left( (1 - \delta)(e^t - 1) \frac{\partial u}{\partial t} + \delta u \right) \\ &\quad + \langle \tilde{\omega}_t, \delta\omega - (1 - e^{-t})(\omega - \omega_\infty) \rangle + \frac{\partial u}{\partial t} - n\delta. \end{aligned}$$

---

<sup>4</sup>This is only going to cause difference for the evolution equations at the level of metric potential similar to that between (1.4) and (2.1).

<sup>5</sup>In fact, one only needs to take care of the case  $\epsilon = 0$  to achieve this.

Consider the minimum value point of the term  $(1 - \delta)(e^t - 1)\frac{\partial u}{\partial t} + \delta u$ . If it is not at the (new) initial time, then at that point, we have

$$\begin{aligned} \langle \tilde{\omega}_t, \delta\omega - (1 - e^{-t})(\omega - \omega_\infty) \rangle &\geq n \cdot \left( \frac{(\delta\omega - (1 - e^{-t})(\omega - \omega_\infty))^n}{\tilde{\omega}_t^n} \right)^{\frac{1}{n}} \\ &= n \cdot \left( \frac{(\delta\omega - (1 - e^{-t})(\omega - \omega_\infty))^n}{e^{\frac{\partial u}{\partial t} + u}\Omega} \right)^{\frac{1}{n}} \quad (2.4) \\ &\geq Ce^{-\frac{1}{n}\frac{\partial u}{\partial t}} \end{aligned}$$

where  $u \leq C$  is applied in the last step, and so one conclude

$$C \geq Ce^{-\frac{\partial u}{\partial t}} + \frac{\partial u}{\partial t},$$

which gives  $\frac{\partial u}{\partial t} \geq -C$  at that point. So  $(1 - \delta)(e^t - 1)\frac{\partial u}{\partial t} + \delta u \geq -C$  at that point in sight of the lower bound of  $u$  from previous argument. We conclude that

$$(1 - \delta)(e^t - 1)\frac{\partial u}{\partial t} + \delta u \geq -C$$

for the space-time, and so

$$\frac{\partial u}{\partial t} \geq -\frac{C}{e^t - 1}.$$

Remember that the time has been translated.

So far, we have obtained the  $L^\infty$ -bounds for both  $u$  and  $\frac{\partial u}{\partial t}$  locally away from the initial time. Only the upper bound of  $u$  is uniform for all time.

The second and higher order estimates can be carried through as in Subsection 3.2 of [2] because the translation of time would make the background form Kähler. Hence we conclude that the weak flow defined in Section 1 becomes smooth instantly. The proof of Theorem 1.2 is thus finished.

**Remark 2.2.** *The situation as  $t \rightarrow 0^+$ , which is indeed the only "weak" spot of the flow, needs further consideration just as in [2]. Since most estimates achieved up to this point are only local away from the initial time, the control of the situation near 0 at this moment is very weak. In fact, strange things can happen. For example,  $[\omega_0]$  might have 0 volume (being collapsed), but the volume would becomes positive instantly.*

### 3 Uniform Estimates up to Initial Time

In this part, we look to achieve some estimates uniform for small time, i.e. for  $t \in (0, T)$  without degeneration towards  $t = 0$ .

The first thing comes to mind would be the uniform  $L^\infty$ -estimate for the metric potential,  $u$ , up to  $t = 0$ . With the Kołodziej type of estimates and even more elementary relation between  $\frac{\partial u}{\partial t}$  and  $u$ , naturally one wants to control the

volume form or  $\frac{\partial u}{\partial t}$  up to the initial time. In order to apply the known results from pluripotential theory to get  $L^\infty$ -estimate, it is natural to require  $[\omega_0] = [\omega]$  to be semi-ample. We focus on the case of  $[\omega]$  being big and semi-ample in this work.

Also, as in [2], it is reasonable to put some restriction on the initial measure,  $\omega_0^n = (\omega + \sqrt{-1}\partial\bar{\partial}v)^n$ . If one just apply the general regularization of the current  $\omega_0$  (in [1] for example), it may not be the case that the volume form of the smooth approximations would have the same kind of control for the Monge-Ampère measure.

Fortunately, in sight of the discussion in Section 1 on the uniqueness of the weak flow, we can make proper choice of the approximation maintaining the measure control and also having the weak flow as limit. More precisely, suppose the measure  $(\omega + \sqrt{-1}\partial\bar{\partial}v)^n$  (assumed to be  $L^1$  to begin with) has some kind of bound (for example, *upper*, *lower* or  $L^{p>1}$ -bounds), then one can use standard process involving partition of unity and convolution to construct a sequence of smooth volume forms,  $\Omega_\epsilon$ , having the same kind of bound as for  $(\omega + \sqrt{-1}\partial\bar{\partial}v)^n$  uniformly and converges to it as  $\epsilon \rightarrow 0$  in  $L^1$  space. Then we can solve the equations  $(\omega + \epsilon\omega_1 + \sqrt{-1}\partial\bar{\partial}v_\epsilon)^n = C_\epsilon\Omega_\epsilon$  where  $C_\epsilon$  are well controlled and tends to 1 after requiring the total measure of  $(\omega + \sqrt{-1}\partial\bar{\partial}v)^n$  being positive. Taking proper normalization for  $v_\epsilon$ , it would decrease to  $v$  as  $\epsilon \rightarrow 0$  from Kołodziej's argument as discussed in 9.6.2 of [13]. The solutions would form a desirable smooth approximation for the construction of weak flow.

With all the above preparation, we look at the situation as  $t \rightarrow 0^+$  for some cases of interests. We are still working directly on the approximation flows, i.e. (1.3) and (1.4), while omitting  $j$  and  $\epsilon_j$  for simplicity.

### 3.1 Volume Upper Bound for Semi-ample and Big $[\omega_0]$

Let's assume the degenerate initial class  $[\omega_0] = [\omega]$  is semi-ample and big. By the generalization of Kołodziej's  $L^\infty$ -estimate (in [4], [12] and [3]), it's enough to have measure  $L^{p>1}$  bound uniform up to the initial time in order to conclude uniform  $L^\infty$ -bound for  $u$ . In order to see this, one needs to notice that although  $\omega_t$  is changing, if we can deal with the most degenerate one,  $\omega$ , then  $\omega_t$  can be treated as  $\alpha\omega + \Phi_t$  where  $\alpha \in (0, 1)$  and  $\Phi_t > 0$  as described in [11].

Now let's search for the upper bound of  $\frac{\partial u}{\partial t}$  uniform up to the initial time under the assumption that the volume of the initial current,  $\omega_0$  has a uniform upper bound, i.e.  $L^\infty$ -bound. The point is to see whether it would be enough to imply the necessary measure bound for small time.

The following equation comes from standard manipulation of (1.4),

$$\frac{\partial}{\partial t} \left( (e^t - A) \frac{\partial u}{\partial t} - Au \right) = \Delta_{\tilde{\omega}_t} \left( (e^t - A) \frac{\partial u}{\partial t} - Au \right) + An - \langle \tilde{\omega}_t, \omega - (1 - A)\omega_\infty \rangle \quad (3.1)$$

for a constant  $A$  to be fixed shortly. Since the flow is driving the class into the Kähler cone, it would take  $A \geq 1$  to give a desirable sign for the last term on



the right hand side of this equation. Unfortunately, Maximum Principle would then give the desirable wrong direction of control for  $\frac{\partial u}{\partial t}$  up to  $t = 0$ .

Indeed one needs to choose some constant  $A < 1$ . Because  $[\omega]$  is semi-ample and big, we can choose  $\omega \geq 0$  and a effective divisor (i.e holomorphic line bundle)  $E$  with the defining holomorphic section  $\sigma$  and a Hermitian metric  $|\cdot|$  such that for any sufficiently small  $\lambda > 0$ ,

$$\omega_0 + \lambda\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2 > 0.$$

**Remark 3.1.** *The estimation of  $\frac{\partial u}{\partial t}$  under discussion would work for  $[\omega]$  being nef. and big. The only difference is one might not have  $\omega \geq 0$ , and so the Hermitian metric would depend on the choice  $\lambda$ . We are mainly interested in the consequential control of  $u$ . That's why semi-ampleness is assumed at this moment.*

Now we can reformulate (3.1) as follows

$$\begin{aligned} \frac{\partial}{\partial t} \left( (e^t - A) \frac{\partial u}{\partial t} - Au + \lambda \log|\sigma|^2 \right) &= \Delta_{\tilde{\omega}_t} \left( (e^t - A) \frac{\partial u}{\partial t} - Au + \lambda \log|\sigma|^2 \right) + \\ &\quad + An - \langle \tilde{\omega}_t, \omega + \lambda\sqrt{-1}\partial\bar{\partial}\log|\sigma|^2 - (1 - A)\omega_\infty \rangle. \end{aligned}$$

By choosing  $A < 1$  according to the size of  $\lambda$ , one can make sure the last term on the right has a definite sign for the small time interval.

Recall that we assume  $\frac{\partial u}{\partial t}|_{t=0} \leq C$ . More precisely, this means  $\omega_0^n \leq C \cdot \Omega$ . We'll keep using the understanding at the beginning of this section.

Applying Maximum Principle, we have

$$(e^t - A) \frac{\partial u}{\partial t} - Au + \lambda \log|\sigma|^2 \leq \max_{X \times \{0\}} (1 - A) \frac{\partial u}{\partial t} + \lambda \log|\sigma|^2 + C \leq C. \quad (3.2)$$

So one arrives at the following degenerate upper bound

$$\frac{\partial u}{\partial t} \leq \frac{-\lambda \log|\sigma|^2 + C}{e^t - A}. \quad (3.3)$$

If we can have the positive number  $\frac{\lambda}{1-A}$  small enough, then the desired  $L^p$ -bound for the measure  $e^{\frac{\partial u}{\partial t}} \Omega$  can be achieved<sup>6</sup>. Notice that the constants  $A$  and  $\lambda$  are related. Although the constant  $C$  may also change with them, that doesn't bring any trouble. This is indeed an assumption on the geometry of the (effective and Kähler) cones.

This degenerate upper bound (3.3) gets better for large  $t$ . In fact, we can see in (3.2) that the assumption on the initial value can be weakened to

$$\frac{\partial u}{\partial t} \Big|_{t=0} \leq \frac{-\lambda \log|\sigma|^2 + C}{1 - A} \quad (3.4)$$

and  $\omega_0^n$  is  $L^1$ .

---

<sup>6</sup>Remark 2.1 describes why one can ignore the term  $e^u$  in the measure.

**Proposition 3.2.** *In the setting of Theorem 1.2, if  $\omega_0$  has  $L^1$  measure satisfying (3.4) representing a nef. and big class, then the weak flow defined in Theorem 1.2 would satisfy (3.3).*

**Remark 3.3.** *If  $\omega$  is a Kähler metric (as in the case discussed in [2]), then one doesn't need to involve the term  $\log|\sigma|^2$  in the above estimation. The measure would have uniform  $L^\infty$ -norm for short time.*

Although this result is not quite satisfying for the sake of the  $L^\infty$  control for  $u$ , it does tell something about the convergence of the weak flow as  $t \rightarrow 0^+$  back to the initial current. We illustrate this as follows.

The degenerate upper bound of  $\frac{\partial u}{\partial t}$  means locally out of  $\{\sigma = 0\}$ <sup>7</sup>,  $u$  is decreasing up to a term like  $Ct$  as  $t \nearrow$ . So by the classic result on weak convergence (as summarized in [5]), for the weak flow,

$$\tilde{\omega}_t^j \rightarrow \omega_0^j \text{ weakly over } X \setminus \{\sigma = 0\} \text{ as } t \rightarrow 0^+, j = 1, \dots, n.$$

Then we can conclude the weak convergence over  $X$  for Monge-Ampère measure using the global cohomology information. Take a sequence of strictly increasing sets exhausting  $X \setminus \{\sigma = 0\}$ ,  $\{U_k\}$  with smooth functions  $\{\rho_k\}$  supported on  $U_{k+1}$  and equal to 1 over  $U_k$ . For any non-negative smooth function  $G$  over  $X$ , we have

$$\lim_{t \rightarrow 0^+} \int_X \rho_k \cdot G \cdot \tilde{\omega}_t^n = \int_X \rho_k \cdot G \cdot \omega_0^n.$$

Now we take supreme of the above convergence to have

$$\sup_k \left( \lim_{t \rightarrow 0^+} \int_X \rho_k \cdot G \cdot \tilde{\omega}_t^n \right) = \sup_k \left( \int_X \rho_k \cdot G \cdot \omega_0^n \right) = \int_X G \omega_0^n.$$

In the mean time,

$$\begin{aligned} \underline{\lim}_{t \rightarrow 0^+} \int_X G \cdot \tilde{\omega}_t^n &= \underline{\lim}_{t \rightarrow 0^+} \sup_k \left( \int_X \rho_k \cdot G \cdot \tilde{\omega}_t^n \right) \\ &\geq \sup_k \left( \lim_{t \rightarrow 0^+} \int_X \rho_k \cdot G \cdot \tilde{\omega}_t^n \right). \end{aligned}$$

So we arrive at

$$\underline{\lim}_{t \rightarrow 0^+} \int_X G \cdot \tilde{\omega}_t^n \geq \int_X G \omega_0^n.$$

The  $=$  has to hold for  $G \equiv 1$ , and so it is not hard to justify it for any test function  $G$ .  $\underline{\lim}$  can be treated in the same way. Hence we conclude

$$\tilde{\omega}_t^n \rightarrow \omega_0^n \text{ weakly over } X \text{ as } t \rightarrow 0^+.$$

**Corollary 3.4.** *In the setting of Proposition 3.2,  $\tilde{\omega}_t^n$  converges weakly over  $X$  back to the initial current  $\omega_0^n$ .*

**Remark 3.5.** *The proof of this corollary can be applied for any wedge power with  $G$  replaced by proper power of a Kähler metric. The conclusion is weaker than that of [2] for the case there.*

---

<sup>7</sup>This can be improved to be the stable base locus set of  $[\omega]$ .

### 3.2 Volume Lower Bound

The consideration of lower bound of volume for small time might look a little strange, but it gets what we want in a more elementary way. There is also an interesting application provided at the end.

Suppose the initial data has a positive volume lower bound, i.e.

$$\left. \frac{\partial u}{\partial t} \right|_{t=0} \geq -C.$$

Together with the nef. assumption for the main theorem, this says more or less that  $[\omega_0]$  is also big. This lower bound can be preserved for the approximation described at the beginning of this section.

Recall the equation (3.2) appeared before

$$\frac{\partial}{\partial t} \left( (e^t - A) \frac{\partial u}{\partial t} - Au \right) = \Delta_{\tilde{\omega}_t} \left( (e^t - A) \frac{\partial u}{\partial t} - Au \right) + An - \langle \tilde{\omega}_t, \omega - (1 - A)\omega_\infty \rangle.$$

Now one choose a proper constant  $A > 1$  so that the last term

$$\langle \tilde{\omega}_t, \omega - (1 - A)\omega_\infty \rangle > 0$$

which is possible from the cohomology picture. Maximum Principle then gives

$$(e^t - A) \frac{\partial u}{\partial t} - Au \leq \max_{X \times \{0\}} (1 - A) \frac{\partial u}{\partial t} + C \leq C$$

in sight of the lower bound of  $\frac{\partial u}{\partial t}$  for the initial time. That is

$$(e^t - A) \frac{\partial u}{\partial t} \leq C.$$

Hence for small time such that  $e^t - A \leq -C < 0$ , we have

$$\frac{\partial u}{\partial t} \geq \frac{C}{e^t - A} \geq -C.$$

This automatically gives lower bound for  $u$  for small time, and also the weak convergence of  $\tilde{\omega}_t^j$  to  $\omega_0^j$  over  $X$  as  $t \rightarrow 0^+$  for  $j = 1, \dots, n$  from the monotonicity of  $u + Ct$ .

**Proposition 3.6.** *In the setting of Theorem 1.2, suppose  $\omega_0^n$  is  $L^1$  and with a uniform positive lower bound, then the metric potential is uniformly bounded for small time and*

$$\tilde{\omega}_t^j \rightarrow \omega_0^j \text{ weakly over } X \text{ as } t \rightarrow 0^+, \quad j = 1, \dots, n.$$

This situation occurs very naturally. For the classic Kähler-Ricci flow, in finite time singularity case, if the singular class  $[\omega_T]$  is semi-ample and big, by parabolic Schwarz Lemma (as in [7]), we have  $\langle \tilde{\omega}_t, \omega_T \rangle \leq C$ . So  $\tilde{\omega}_t^n \geq C\omega_T^n$ .

Here the semi-ample  $[\omega_T]$  generates a map  $P : X \rightarrow \mathbb{CP}^N$  and  $\omega_T = P^*\omega_{FS}$  for the standard Fubini-Study metric  $\omega_{FS}$ . It being big means  $P(X)$  is of the same complex dimension as  $X$ .

If  $F(X)$  is smooth, then the push-forward of  $\tilde{\omega}_T$  would be in the setting of Proposition 3.6, and so the weak flow over  $F(X)$  would weakly converges back to the push-forward current.

*This simple observation makes the picture of global weak flow on complex surface of general type very satisfying.*

**Acknowledgment 3.7.** *The author would like to thank everyone who has supported this work and beyond. The collaboration on a earlier work with Xiuxiong Chen and Gang Tian has provided precious knowledge on this topic and valuable experience. Jian Song and other people's interest and discussion are also important. The very recent work of Jian Song and Gang Tian ([8]) considers this problem in general algebraic geometry setting. Their results are of different flavor from ours. Finally one can not say enough to the friendly environment of the mathematics department at University of Michigan, at Ann Arbor.*

## References

- [1] Błocki, Zbigniew; Kołodziej, Sławomir: On regularization of plurisubharmonic functions on manifolds. Proc. Amer. Math. Soc. 135(2007), no. 7, 2089–2093 (electronic).
- [2] Chen, Xiuxiong; Tian, Gang; Zhang, Zhou: On the weak Kähler-Ricci flow. arXiv:0802.0809 (math.DG) (math.AP). To appear at Transactions of the American Mathematical Society.
- [3] Demailly, Jean-Pierre; Pali, Nefton: Degenerate complex Monge-Ampère equations over compact Kähler manifolds. arXiv:0710.5109 (math.DG).
- [4] Philippe Eyssidieux; Vincent Guedj; Ahmed Zeriahi: Singular Kähler-Einstein metrics. ArXiv, math/0603431.
- [5] Kołodziej, Sławomir: The complex Monge-Ampere equation and pluripotential theory. Mem. Amer. Math. Soc. 178 (2005), no. 840, x+64 pp.
- [6] Kołodziej, Sławomir: Hölder continuity of solutions to the complex Monge-Ampere equation with the right hand side in  $L^p$ . ArXiv, math.CV/0611051.
- [7] Song, Jian; Tian, Gang: The Kähler-Ricci flow on minimal surfaces of positive Kodaira dimension. Invent. Math. 170 (2007), no. 3, 609-653.
- [8] Song, Jian; Tian, Gang: The Kähler-Ricci flow through singularities. arXiv:0909.4898 (math.DG).
- [9] Tian, Gang: New results and problems on Kähler-Ricci flow. To appear in Asterisque.

- [10] Tian, Gang: Geometry and nonlinear analysis. Proceedings of the International Congress of Mathematicians, Vol. I (Beijing 2002), 475–493, Higher Ed. Press, Beijing, 2002.
- [11] Tian, Gang; Zhang, Zhou: On the Kähler-Ricci flow on projective manifolds of general type. Chinese Annals of Mathematics - Series B, Volume 27, a special issue for S. S. Chern, Number 2, 179–192.
- [12] Zhang, Zhou: On degenerate Monge-Ampere equations over closed Kähler manifolds. International Mathematics Research Notices 2006, Art. ID 63640, 18 pp.
- [13] Zhang, Zhou: Degenerate Monge-Ampere equations over projective manifolds. Mathematics Ph. D. Thesis at MIT.